III. Mathematical Thinking

Background and Criteria

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This domain's focus is on children's approaches to mathematical thinking and problem solving. Emphasis is placed on how students acquire and use strategies to perceive, understand, and solve mathematical problems. Mathematics is about patterns and relationships and about seeking multiple solutions to problems. In this domain, the content of mathematics (concepts and procedures) is stressed, but the larger context of understanding and appreciation (knowing and doing) is also of great importance.

Note: Preschool-3 performance indicators are noted unless the indicator starts at a higher grade. In those circumstances, the performance indicator is written starting at the lowest grade with the grade level noted in parentheses.

A. Processes and Practices
1. Shows interest in solving problems.

Problem solving is an essential mathematical process and practice. To develop proficiency in mathematics, competence in problem solving is "interwoven and interdependent" on conceptual understanding, procedural fluency, adaptive reasoning, and productive dispositions. (National Research Council [NRC], 2001, p. 116). In other words, problem solving is the primary application of the math concepts and skills taught to young children. All recent standard documents (Common Core, 2010; National Council Teachers of Mathematics [NCTM], 2006; National Association for the Education of Young Children [NAEYC] and NCTM, 2002) include problem solving as a standard, guideline, or an overarching principle.

For very young children, the research indicates that children are interested in mathematics and use it to make sense of their physical and social world (NRC, 2009). Children begin making sense of problems at an early age which is critical for problem solving. Common Core (2010) states, "a lack of understanding (i.e., if it doesn't make sense) effectively prevents them from the practices (e.g., problem solving)."

While most problems for children involve numbers, there are also spatial or geometric problems. Whatever the type, children will be most able to solve problems that are connected to their experiences and then move from familiar to less familiar. With guidance, young children can begin to *mathematize story situations* (NRC, 2009, p. 44). Stories provide wonderful opportunities for problem solving. (Clements & Sarama, 2009, p. 206). In other contexts, children often solve math problems during play. Play tools, experiential play centers, or materials specifically used to develop math concepts are used by children to act out problems or develop problem-solving strategies. In addition, when children play with math concepts, they are the solvers and they can demonstrate the application of math concepts as they play (Ginsburg, Inoue, and Seo, 1999; Holton, Ahmed, Williams, & Hill, 2001).

Children use a variety of strategies to solve problems and more flexible strategies evolve as they have more experience. Young children experiment by using objects, fingers, and drawings. They most often rely on concrete objects or pictures to help them conceptualize and solve a problem (NRC, 2009). In number and operations, children first use counting to solve the problem, then direct modeling strategies, and finally number facts (both derived and recalled). As children grow older, they begin to use symbols and numbers to help them solve quantitative problems. Carpenter, Fennema, Franke, Levi, & Epson (1999) found that older children solve multidigit problems first by counting by ones, then counting by tens, and then use a direct algorithm to solve place value problems. Interaction with peers is also important and children who talk about the problems with others can use their feedback as a strategy. Finally, research also indicates that persistence is an important component to problem solving, and even very young children can persist at a problem, if they are interested in it and it is challenging (Copley, 2008). However, if children exhaust their ability to work through a problem, they may not persist in solving it. As a result, additional problem solving strategies also need to be developed and taught.

2. Begins to reason quantitatively.

Mathematical reasoning must be at "the center of mathematics learning" and when paired with problem solving, is often called the "heart" of mathematics (Russell, 1999, p. 1; NCTM, 2006.) The Common Core Practice Standards list quantitative reasoning as one of the practice standards because "mathematically proficient students make sense of quantities and their relationships in problem situations." Other studies stress the importance of being able to use adaptive reasoning and approach problems in alternative ways in order to solve math problems (NRC, 2001, p 129, NAEYC and NCTM, 2002). Quantitative reasoning specifically applies to numbers and their operations. Research states that children who understand the meaning of different operations and who can see the relationship between quantities given in word problems with those meanings best demonstrate quantitative reasoning (Sowder, 1988). In fact, rather than using key words for word problems, researchers have found that it is more important that children understand the quantities involved and their relationships to each other. (Diezmann & English, 2001). The research further finds that if children can develop their ability to reason mathematically, they will begin to note patterns or regularities in the world and make the connections necessary for powerful mathematics (NRC, 2009).

Some researchers think that children's reasoning ability is quite limited until they are in upper elementary or middle school. However, when children are asked to talk about how they arrived at their solutions to problems, children as young as ages 4 and 5 display evidence of reasoning behaviors (NRC, 2001, p. 129). Current research indicates that young children can reason; however, to evaluate the adequacy of the child's reasoning, we must understand "where the child is coming from." We must appreciate the child's reasoning, and we cannot unless we understand the child's perspective and their sense-making experiences. (Tang & Ginsburg, 1999, p. 48).

To help with reasoning and understanding, children need to use appropriate tools strategically. For example, rulers are appropriate once children can compare lengths indirectly (i.e., they understand transitivity). If, on the other hand, they only can compare lengths directly (generally, ages 3, 4, and 5), the accurate use of rulers would be inappropriate. Likewise, the use of a calculator would not be appropriate for a third grader who was adding 10 to 23, but it would be very appropriate and strategic for a first grader who wanted to see if there was a pattern when 10 was added to any number. Place value blocks are another appropriate strategy when, for example, second graders are adding or subtracting three-digit numbers; however, Unifix® cubes used to make ten rods would be more appropriate for kindergartners who are learning that 10 is composed of 10 units. In conclusion, "considering their minimal experience, young children are impressive problem solvers. They are learning to learn and learning the rules of the 'reasoning game'" (Clements & Sarama, 2009, p 204).

3. Uses words and representations to describe mathematical ideas.
Representing is central in mathematics. A representation is typically a sign, character, or object and can symbolize, depict, encode, or represent something other than itself. In the realm of early childhood mathematical representation and communication, the child's drawings or pictures are "tools" for understanding mathematics and a "language" for sharing mathematical work (Woleck, 2001). Drawings and all other sign systems are built on a foundation of verbal speech. In other words, oral language precedes and is a bridge to graphic representation (Vygotsky, 1978).

Research indicates that 4-year-olds represent quantities globally; they draw vague pictures. They then progress by drawing recognizable pictures, using one-to-one correspondence with symbols and then numbers. Generally, by the time they finish kindergarten, they use cardinal values of the objects and write and label the number of objects (Kamii, Kirkland, & Lewis, 2001, p. 28–29). With experience and guidance, young children can begin to create increasingly more abstract representations and model operations with expressions and equations (NRC, 2009).

Fingers are often used by young children to count or perform simple operations. Researchers have found that finger representations accelerate children's single-digit addition and subtraction as much as a year, more so than methods in which children just count objects or pictures. Drawings and diagrams that children produce are also important representational tools. Those who are best at solving problems with objects, fingers, or counting are least likely to use those less sophisticated strategies in the future (Clements & Sarama, 2009, p. 68).

4. Begins to recognize patterns and makes simple generalizations. (Preschool-4)

Algebra is the fundamental language of mathematics and has been advocated as essential for all (Silver 1997, p. 206). Algebraic reasoning involves generalizing from patterns and developing a general mathematical statement about patterns, structures, properties, or relationships. For young children, the identification of patterns and structure is foundational to mathematical thinking (and later progresses to algebraic thinking). Structure is important at all levels in mathematics. Pattern identification is part of a foundation for later algebraic thinking (NRC, 2009).

It is important for young children to identify number patterns and a way of thinking about mathematical structures that encourage them to make generalizations. This type of thinking is much more than making simple linear color patterns. Clements and Sarama (2009, p 192) assert, "Children who do not develop this type of knowledge tend to make little progress in mathematics." In the elementary years, children benefit from describing patterns with numbers. Even before first grade, children begin to identify a counting pattern when counting past 20. Research also indicates that instruction that encourages algebraic thinking has been shown to enhance students' learning of arithmetic (Carpenter, Franke, & Levi, 2003). For example, children who learn patterns such as the associative property of addition (i.e., you can add in any order and still get the same answer) are also learning about the simplest forms of algebra. Mason (2008) argues that children bring natural powers of generalizing to the elementary classroom in many nonmathematical contexts. The goal of early mathematics should be to enlist those powers for number and visualization so that children begin to generalize as a mathematical activity.

B. Number
1. Shows interest in counting.

Young children typically learn to count as one of the earliest indicators of "math learning." Researchers (Gelman & Gallistel , 1978; Baroody, 1992) identified five counting principles: 1) the word number list must be said in the same order to count accurately; 2) each item must be counted using one-to-one correspondence; 3) the last number in the count sequence answers the "How many?" questions; 4) things to be counted are discrete items; and 5) items can be counted in any order. Verbal (rote counting) and object counting are both important to the ability to count accurately and consistently. Developmentally, children first verbally count by reciting the numbers 1 to 10 and keeping one-to-one correspondence between the words and objects for small numbers. By age 4, they accurately count objects 1 to 5 in arrangements and answer the "How many?" question with one number. Four-year-olds are also counting objects to 5 and are developing verbal counting skills to 20. Kindergartners count objects accurately to 30, backward from 10 to 1, and recognize counting errors. Six-year-olds generalize the counting pattern and are able to count to 100 as well as count on or back from a specific number. Counting by 5s and 10s is learned as children learn to group in tens and begin to count money and tell time in first grade (Clements & Sarama, 2009; NRC, 2009).

2. Shows interest in quantity.

Understanding numbers includes concepts of quantity and relative quantity. Children begin to develop an understanding of quantity as early as age 2 when they physically see objects and say "two" or hold up two fingers to identify how old they are. Subitizing is the process of identifying the number of items in a small set or cardinality. As children get older (ages 4 or 5), they begin to conceptually subitize, a process where they decompose a number into two parts and are able to tell its cardinality quickly without counting (Clements & Sarama, 2009). Children in first, second, or third grade need to see large quantities as bundles or groups of tens, hundreds, and thousands to understand their quantity. Whatever the age, a variety of visual quantitative models and representations are important for understanding numbers. Children can use ten-frames, hundreds boards, number lines, place value blocks, or tally marks to represent quantity (NRC, 2001).

Comparing is a common activity for children along with counting (see B1) and addition and subtraction (see C1).. Three-year olds-compare groups of objects visually and label a quantity as "more" because it looks like it is or it takes up more room. Later, children compare by matching and still later, when number order is learned, they compare collections of objects by counting (i.e., numbers with higher cardinalities have more). At the first-grade level, children can tell which collection has more and also subtract to answer the "How many more?" question (NRC, 2009). In later grades, children compare number quantities (e.g., or the prices of ice cream cones or amounts in a collection) by analyzing the digit in the largest place and comparing the digits using their understanding of place value. They use mathematics symbols (<, >, =, or not equal) to illustrate the relationship between quantities (NRC, 2001).

3. Begins to estimate quantity. (Kindergarten)

Estimation involves sense-making and quantitative reasoning. Children in kindergarten can estimate quantities (or numerosity estimation) by estimating in ranges such as more than 10, more than 20, or fewer than 30. Students who can look at a grouping of objects, estimate a range of possible numbers, and explain their strategies of estimation are developing quantitative reasoning, not guessing (Clements & Sarama, 2009). Computation estimation can be used in later grades as children begin to develop computational accuracy and understanding. Estimating before solving a problem can facilitate number sense, reasoning, and place-value understanding by encouraging students to generate approximate results. In other words, they estimate first to find a reasonable response, calculate the problem, and then check to see if the exact result is reasonable by comparing it with the estimate. (In reality, many students, in fear of being wrong, find the exact answer first and then round it as their estimation—a method that defeats the whole purpose.) Estimating is a practical, rather complex skill (Sowder & Wheeler, 1989). It may ask students to restructure the problem, by rounding, compensation, or a method that makes sense to them and mentally complete the operation. It also requires recognizing that the appropriateness of an estimate is related to the problem and its context (NRC, 2001).

3. Understands fraction concepts. (Third Grade)

The National Mathematics Advisory Panel (2008) identified proficiency with fractions as a major goal for Pre-K through 8 mathematics education; "such proficiency, …at the present time, is severely underdeveloped" (p. xvii). To address this issue, research recommends that students in first and second grade develop initial fraction concepts from their informal understanding of sharing and by representing the whole partitioned into equal parts (Empson, 2002; Steffe & Olive, 2002; Witherspoon, 2002). The Common Core State Standards (CCSSO, 2010) further recommend that third graders develop an understanding of fractions, beginning with unit fractions (1/a). Students can view fractions as being built out of unit fractions, and use fractions along with visual fraction models to represent parts of a whole. To have a strong foundation for understanding fractions, students must understand that the size of a fractional part is relative to the size of the whole. They should primarily solve problems that involve comparing fractions by using visual fraction models (NCTM, 2006).

C Operations and Algebraic Thinking
1. Begins to understand addition and subtraction.

Addition and subtraction are inverse operations that span across all age levels and elementary grades. The NRC (2009) outlines four beginning steps to addition and subtraction and assigns typical ages for their completion. In Step One (typically ages 2 to 3), children can solve situation and oral number word problems with totals less than or equal to 5. In Step Two (typically age 4), children can use conceptual subitizing (see B2) and cardinal counting to solve situation, word, and oral number problems with totals less than or equal to 8. They also solve numerical situations by modeling actions with objects and fingers and mentally, visually, or count out the answer. In addition, they can learn the "partners" for numbers 3, 4, and 5 (e.g., 2 and 3 are partners to make 5). In Step Three (typically kindergarten), children use conceptual subitizing and cardinal counting to solve situation, word, oral, and written numeral problems with totals less than or equal to 10. For oral or written numerical problems, they use fingers, objects, or a drawing to solve the problem. In Step Four (typically first grade), children count on or use a derived fact method for problems with totals less than or equal to 18. In Grades 2 and 3, children should begin to develop, use, and discuss procedures to solve addition and subtraction calculation problems. As they understand the base-ten system (see C5) students may use invented algorithms and eventually they can employ a variety of algorithms to calculate solutions to addition and subtraction problems. The traditional written method may be the most beneficial to students. However, there are many alternative methods which research indicates may be more understandable especially because they specifically emphasize place value. Using methods that build on children's thinking and early experiences help students use written algorithms with meaning (Carroll & Porter, 1998).

2. Demonstrates basic number combination and computational fluency. (Kindergarten)

Fluency is demonstrated when mathematical solutions are "accurate and solved (fairly) rapidly and (relatively) effortlessly with a basis of understanding that can support flexible performance when needed" (NRC, 2009, p. 128). Fluency has been advocated by many national groups and recommendations for fluency "endpoints" based on research are fairly consistent across each of them, specifically, add/subtract within 5 for kindergarten, add/subtract within 10 for first grade, add/subtract within 20 and add/subtract within 100 using pencil and paper for second grade, multiply/divide within 100 and add/subtract within 1000 for third grade (NCTM, 2006; NRC, 2001; CCSSO, 2010).

Fluency most often deals with what some call "memorizing basic facts." Research indicates that is a misleading idea. Rather, the "facts" should be thought of as arithmetic combinations that are learned when arithmetic concepts form an organizing framework and are stored for easy access. Research also shows that fluency is developed over time. In a recent study, first graders memorized the arithmetic combinations for addition and subtraction and then received little support in second grade. The result was that less than 26% of the children in this large scale study were able to demonstrate fluency at the end of second grade. Also, fluency is not developed through the use of timed tests. Researchers concluded that memorizing without understanding does not help develop fluency. Rather, using strategies like "Make-a-Ten" or using derived combinations together with memorization are more likely to develop fluency (Sarama & Clements, 2009).

3. Begins to understand the base ten system (place value). (Kindergarten)

"Our decimal system is versatile and simple, although not necessarily obvious or easily learned," (NRC, 2001, p. 96). Research indicates that children often have a weak understanding of place value and identify the value of a digit in the multidigit number 5267 as simply "2" rather than "200." Language plays an important role in the understanding of place value. Unlike Asian languages, the English verbal counting patterns of the numbers, 11, 12, 13, etc., do not help children understand ten, rather than 11 it could be "ten-one" or rather than 15 it could be "ten-five." Other number words present problems as well, rather than 20 or 30, it could be "two tens" or "three tens." The superiority of Asian students in understanding grouping procedures indicates that language combined with experiences in grouping number in sets of ten, regrouping, or trading would be helpful as children begin to understand place value (Sarama & Clements, 2009).

An understanding of place value is necessary for multidigit operations. Researchers have identified five levels of children's understanding of place value: 1) 36 is viewed as the number that comes after 35 and the digit 3 is simply a 3; 2) 36 is viewed as 30 and 6, but may be written as 306; 3) 36 is viewed as counting by tens and then ones, so 36 would be 10, 20, 30, 31, 32, 33, 34, 35, and 36; 4) 36 is described using English words, one 10, two 10s, three 10s, 1, 2, 3, 4, 5, 6; and 5) an integrated place value understanding which allows them to use flexible strategies to solve multidigit problems (Fuson, Wearne, Hiebert, Murray, Human, Olivier, et al., 1997).

4. Shows beginning understanding of multiplication and division. (Second Grade)

Children progress through a sequence of multiplication procedures similar to those of addition. They understand multiplication by making groups of equal size or counting the squares in arrays with a number of rows and columns. To solve, they can count all or skip count using patterned lists (e.g., the list for multiplication by 7 is 7, 14, 21, 28, 35, 42, etc.). As with addition and subtraction, children invent their own thinking strategies that allow them to use what they know and derive other products. When they learn to multiply by 10s or 100s, the patterns they notice help them make generalizations about products. There are 100 combinations that children must learn for single-digit multiplication. Finding patterns and making generalizations about the properties of multiplication (see A4) are the most effective strategies for understanding and learning these combinations (NRC, 2001). Please note that young children can overgeneralize about multiplication and division and believe that multiplication makes larger numbers while division makes smaller numbers. While this is true for most whole numbers, it is not true for rational numbers less than one, zero, or one (Graeber, 1993).

Multiplication and division are inverse operations and are of three types of problems: 1) Multiplication – two factors are known, product is unknown; 2) Measurement Division – the whole is known, the number for each group is known, and the number of groups is unknown; and 3) Partitive Division – the whole is known, the number of groups is known, and the number in each group is unknown. Similar to addition and subtraction, children solve these problems using direct modeling strategies and then counting to solve (Carpenter, et al., 1999).

D. Measurement
1. Shows understanding of some comparative words.

Measurement is an important topic to young children; they love to compare everything and words like "bigger" and "more" are natural parts of their vocabulary. In mathematics, measurement is connected to both number and geometric concepts. In addition, the early work of Piaget and his collaborators advocated that conservation was a constraint on children's ability to measure. More recent studies have "generally failed to support the contention that there is a tight coupling between understanding a spatial measure and knowing when it is conserved" (NRC, 2001, p. 201).

Identifying measurable attributes of objects or events (for time) is an important first step in measuring. Length (how long, how wide, how tall, how short), area (how much is covered), weight (how heavy, how light), volume (how much does it hold), and time (how long does it take?) are all important, measurable attributes that children can compare, order, and describe. Typically, because of children's development, the child compares two objects or events first and then progresses to comparing more and putting them in order. Children as young as age 3 can identify length as an attribute, 4-year-olds can physically align two objects and compare them to a third, and 5-year-olds can begin ordering one to six lengths. Regarding area measurement, 4-year-olds will compare areas using only one side of figures, thus ignoring width and focusing on the length. Five-year-olds can begin to count squares that cover a particular area but lose track quickly due to their unsystematic methods. For volume measures, 3-year-olds can identify volume as an attribute, 4-year-olds can compare two containers, and 5–year-olds can order three containers using transitive reasoning [i.e., If A < B, B < C, then A < C] (Clements & Sarama, 2009).

2. Participates in measuring activities.

The measuring process involves many tools, skills, techniques, and specific vocabulary. Research in this area has not been plentiful and recently measurement concepts have been refocused based on new information. NCTM's (2006) *Curriculum Focal points for Prekindergarten through Grade 8 Mathematics,* proposed focal points based on research in an effort to help children "learn deeply" the important ideas of mathematics. In prekindergarten, children identify objects and directly compare them by the attributes of length and weight. In kindergarten, they begin to order objects by length specifically and continue to compare other objects by their measureable attributes. In first grade, length measurement is again strengthened and children begin to iterate using units end-to-end and count the number of units as a measurement. In second grade, children are able to accurately measure lengths and understand the concepts of measurement (e.g., equal-size units, transitivity, inverse relationship in the size of the unit and the number of units, iteration, and partitioning). Then, with an understanding of fractions (see A4), third graders become more proficient and accurate measuring the lengths, perimeters, and areas of objects.

A recent NRC report (2009, p.197) asserted that by the ages of 4 through 5, most children "can learn to overcome perceptual cues and make progress in reasoning about and measuring quantities." They are ready to measure, but a lack of experiences typically postpones that learning until the end of the primary grades. They advocate that the use of standard measuring tools, modeling the measurement process, and purposeful measuring opportunities should be part of every young child's experience.

E. Data Analysis
1. Begins to collect, classify, and represent data. (Kindergarten)

Data analysis (or graphing) has long been a mathematics topic in prekindergarten, kindergarten, and primary grade classrooms, primarily initiated by a teacher. It is listed as a "connection to focal points" by NCTM (2006) and included as a Common Core Curriculum Standard (2010). However, the collection of data and its representation should not be a teacher-centered activity. Both concepts of measurement and number are central to data analysis and, if it is a child-centered activity, it can be effectively connected to children's learning. The collection of data should start with a question of interest to children. Then, children should collect the information and classify it in a way that seems most appropriate to them. This data could then be represented using a "real graph" where the actual objects or representations or children could be placed in one-to-one correspondence so the number of objects in one category could be compared. Other representations of the data could then be translated from these "real graphs" with sticky notes, colored dots, or crayon pictures. Research indicates that the representation of data can be helpful in developing children's mathematical thinking. However, with no discussions or interpretations of the representations, mathematical thinking would not be extended (NRC, 2009).

F. Geometry
1. Shows understanding of several positional words.

Geometry is the study of shapes and space. Spatial sense is related to mathematics competencies and includes two main spatial abilities: spatial orientation and spatial visualization and imagery (Clements, 1999). Developing spatial orientation is connected to specific language. Research indicates that children who were asked to find a hidden object and given specific position words (e.g., in, over, under) were much more able to find the object than children who were given a more general description (e.g., over here). Exposure to spatial language during spatial experiences appears to be useful in learning and developing spatial orientation. Spatial visualization and imagery require that a child mentally keeps a 2- or 3-dimensional shape in mind and is able to reproduce it, rotate or flip it mentally, or match the orientation of the shape. Children between the ages of 4 and 5 had difficulty visualizing when the orientation was different, but children ages 6 and up were able to mentally rotate or flip the object to match the comparison one. Based on many spatial studies, researchers have hypothesized that the differences in spatial ability may largely be the result of experiential differences during early childhood and that preschool programs should foster spatial learning (NRC, 2009).

2. Identifies several shapes.

Geometry is the study of shapes and space. Two- and 3-D shapes or solids have many defining attributes as well as other attributes that do not uniquely define them. The difference between “defining” attributes (e.g., number of sides or edges; shape of faces; equality of line segments, rolls) and “non-defining” attributes (e.g., orientation, color, and size) is important if a shape is to be identified, drawn, manipulated, labeled, measured, and analyzed (Clements, 1999). Recommended by all national and state organizations (NCTM, 2006; CCSSO, 2010), reasoning about shapes and their attributes is an important standard in mathematics.

Developmentally, 4-year-olds begin to recognize shapes by classifying typical circles, squares, and triangles. They generally can’t differentiate sides and corners but usually identify them using “looks like” vocabulary. They can compare two shapes by matching sides and can make a shape that “looks like” a goal shape; however, when they look for differences in shapes, they may only examine part of the shape. Five-year-olds begin to recognize more rectangles by sizes, shapes and orientation and can identify sides as distinct geometric objects. They compare shapes and look for differences by identifying the sides and “corners.” They also recognize most familiar shapes and typical examples of other shapes such as hexagon, rhombus, and trapezoid. Six- and 7-year-olds can identify most shapes by their defining attributes and use correct geometric language to label them. Their “looks like” language transitions to “because it has” a specific attribute language (Clements & Sarama, 2009).

3. Begins to explore composing and decomposing shapes.

Composing and decomposing shapes provides a direct connection between geometry and measurement, including an introduction to fractions at the first grade level (CCSSO, 2010). To build a foundation for measurement of area, volume, congruence, similarity, and symmetry in later grades, research suggests that children should build, draw, and analyze 2- and 3-dimensional shapes (NRC, 2001).

Composing shapes from 3-D shapes provides an excellent opportunity for 3- and 4-year-olds to use trial and error as they create new structures (Copley & Oto, 2000). As they get older, they are able to anticipate what can be produced when they add new blocks to their structures (Clements & Sarama, 2009); in addition, they can also represent what they have made (see A3) and ask their peers to make a similar structure based on a 2-D photo (Copley, 2006–2013). In the lower primary grades, children are able to decompose 3-D shapes as necessary to measure the area of the faces or to calculate how many cubic centimeters a particular cube can hold.

Four-year-olds can compose and decompose 2-D shapes and make pictures in which a shape represents a unique role (like the part of an arm for a pattern block “person”) and the sides of the shapes touch. Five-year-olds fill easy puzzles that suggest the placement of a shape; with experience, they can fill a square with triangles by rotating and flipping them until the square is completely covered. They can learn to rotate and flip the pieces intentionally so that they fit (Clements & Sarama, 2009). In third grade, children are able to decompose a shape or compose a shape to find the area of a triangle, a rectangle, or an irregular shape. This connection between measurement and geometry continues in the later grades.

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